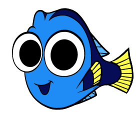
Learning objectives:

- ☐ use quantifiers to quantify a predicate, in which the variable belongs to a set
- ☐ practice with sets, predicates and quantifiers
- ☐ translate English statements to math

Do you speak math?



Usually, the problems you will encounter (not just in this course, but in life) will be stated in *words*, not in the mathematical notation we have introduced earlier. Today, we will combine what we have learned so far (propositions, predicates, sets, quantifiers) and convert some text descriptions into a mathematical representation. The goal for today is to practice as much as possible.

Example 1:

Consider the following statements (Written by Lewis Carroll)

All hummingbirds are richly colored.
No large birds live on honey.
Birds that do not live on honey are dull in color.
Hummingbirds are ____________ .

What can we conclude about hummingbirds?

1 Quantifiers

Quantifiers can be used to quantify the values of predicates. That is, they turn predicates into statements.

The **universal quantifier** is used to express "for all elements in the set", and is represented by $\forall$:

$$\underbrace{\underbrace{\forall x \in \mathbb{N}}_{\text{quantifier}}\ ,\ \underbrace{x > 0}_{\text{predicate}}}_{\text{statement (true)}}$$

The **existential quantifier** is used to represent "an element in the set exists such that ...", and is represented by $\exists$:

$$\underbrace{\underbrace{\exists x \in \mathbb{N}}_{\text{quantifier}}\ :\ \underbrace{x < 0}_{\text{predicate}}}_{\text{statement (false)}}$$

Make sure to include a domain!

It is incorrect to write $\forall x,\ x < 0$ or $\exists x :\ x < 0$. You need to specify the **domain** for x. Is x a real number? An integer? An animal? Correct versions of these statements would be

$$\forall x \in S, x < 0$$
$$\exists x \in S \colon x < 0$$

where S is some set (**domain**).

The truth of quantifiers is summarized in the following table.

Statement	When True?	When False?
$\forall x \in S,\ p(x)$	$p(x)$ is true for every x	there is an x for which $p(x)$ is false
$\exists x \in S\colon p(x)$	there is an x for which $p(x)$ is true	$p(x)$ is false for every x

There are a few important ingredients here. First, is that a quantifier must include a domain on which the variable is quantified. In the example above, the domain is $\mathbb{N}$. Second, do you notice anything different about the universal and existential quantifiers (other than the $\forall$ or $\exists$)? Notice that a universal quantifier ($\forall$) requires a comma to separate the quantifier and predicate, whereas the existential quantifier ($\exists$) requires a colon (:) to separate the quantifier and predicate. This colon should be read as "such that", similar to when we talked about set-builder notation. Again, you may use a vertical bar (|) to represent the "such that" in order to separate the quantifier and predicate.

1.1 Negating quantified expressions

Sometimes you may need to work with both a predicate, and the negation of that predicate, so it's useful to have some identities up our sleeves. Consider the statement:

Every student at Middlebury lives on campus.

which can be seen as $\forall x \in S, p(x)$ with $p(x)$ being the predicate *student x lives on campus*. The negation of this statement is

Not every student at Middlebury lives on campus.

which means that there is (exists!) some student that does not live on campus: $\exists x \in S \colon \neg p(x)$. We can do the same with a statement that relies on existential quantification:

There is a student that lives on campus.

Our predicate $p(x)$ is still *student x lives on campus* and this statement can be seen as $\exists x \in S \colon p(x)$. Again, taking the negation of this statement gives us:

It is not the case that a student lives on campus.

Which means that every student lives off campus. Mathematically, $\forall x \in S, \neg p(x)$. The results of negating quantified statements is summarized on the right.

Negation	Equivalent to . . .
$\neg\forall x \in S, p(x)$	$\exists x \in S \colon \neg p(x)$
$\neg\exists x \in S \colon p(x)$	$\forall x \in S, \neg p(x)$

2 Tips

Before we look at a bunch of examples, here are some tips to keep in mind when translating between English and math. Keep in mind that these are just tips, but there is no universal set of steps to follow when translating sentences into math. The best thing to do is practice!

- you can combine multiple elements of the same type with a quantifier. For example $\forall x, y \in \mathbb{Z},\ p(x, y)$.
- any non-quantified variables should be inputs to a predicate.
- when a variable in a predicate is quantified, rewrite it as a new predicate in terms of the remaining variables. For example, given a predicate $p(x, y)$,

$$\underbrace{\exists y \in S}_{y \text{ is quantified}}\colon p(x, y) \equiv q(x) \quad \text{(is now a predicate in } x\text{)}$$

- all variables in a statement should be quantified

3 Examples

Example 2:
Translate the sentence

Some birds can fly.

into logic.

Solution:
Let the universe be a set that contains all animals, denoted by A. Let $b(x)$ be the predicate that x *is a bird*. Let $f(x)$ be the predicate that x *can fly*. We then have

$$\exists x \in A\colon b(x) \wedge f(x)$$

Warning: It might be tempting to write $\exists x \in A\colon b(x) \implies f(x)$, which would be incorrect. But remember the truth table for an implication ($F \implies F$ is true). So if x is a pig, then $b(x) = F$ and $f(x) = F$ because pigs are neither birds, nor can they fly. The implication would then be true which is an incorrect translation of the original sentence.

Example 3:
Let S be the set of all people. Let $p(x, y)$ be the predicate that x is a parent of y. Determine whether the following statements are true or false:

(a) $\forall x \in S, \exists y \in S\colon p(x, y)$

(b) $\forall x \in S, \exists y \in S\colon p(y, x)$

(c) $\exists x \in S, \forall y \in S\colon p(x, y)$

(d) $\exists x \in S, \forall y \in S\colon p(y, x)$

Solution:
(a) False: this translates to *every person has a child*

(b) True: this translates to *every person has a parent*

(c) False: this translates to *one person is the parent of all people*

(d) False: this translates to *one person is the child of all people*

Example 4:
Let S be the set of people in the class. Let the predicate $F(x, y)$ mean that person x considers person y to be their friend ($x \neq y$). Note that friendship here is **not** inherently mutual. That is, I might consider someone my friend who may not consider me their friend. Translate the English description of each predicate or proposition into a logical formula using quantifiers.

(a) Proposition p states that there is some super likable person in the class that everyone considers their friend.

(b) Proposition q states that everyone in the class has at least one person they consider to be their friend.

(c) Proposition r states that there is a mutual friendship in the class. That is, there are two people that consider each other to be their friend.

(d) Predicate $a(x)$ states that person x considers more people to be their friend than anyone else in the class.

(e) Predicate $b(x, y)$ that everyone who considers person x to be their friend also considers person y to be their friend.

Solution:
(a) $p \equiv \exists x \in S\, \forall y \in S,\ F(y, x)$

(b) $q \equiv \forall x \in S\, \exists y \in S,\ F(x, y)$

(c) $r \equiv \exists x, y \in S \colon F(x, y) \wedge F(y, x)$

(d) $a(x) \equiv \forall z \in S,\ z \neq x \implies |\{y \in S \colon F(x, y)\}| > |\{y \in S \colon F(z, y)\}|$

(e) $b(x, y) \equiv \forall z \in S,\ F(z, x) \implies F(z, y)$

Example 5:
Back to our Lewis Carroll example.

Solution:
Hummingbirds are *small*. Let the predicates be defined as follows:

$p(x)$: *x is a hummingbird*

$q(x)$: *x is large*

$r(x)$: *x lives on honey*

$s(x)$: *x is richly colored*

The statements can be translated into:

$$\begin{aligned} \forall x\,(p(x) &\implies s(x)) \\ \neg\exists x\,(q(x) &\wedge r(x)) \\ \forall x\,(\neg r(x) &\implies \neg s(x)) \\ \forall x\,(p(x) &\implies \neg q(x)) \end{aligned}$$

The conclusion is, therefore, that if x is a hummingbird, then it is not large. In other words, hummingbirds are small. We'll look at how to deduce conclusions like this next class.