Learning objectives:

- ☐ deduce new truths from existing statements using rules of inference
- ☐ use a computational proof-assistant for checking your deductions

We have seen a few examples in which we drew conclusions from a set of existing statements, or used a truth table to verify whether two expressions were equivalent. Today, we'll look at how we can deduce new statements and then verify them with a mini proof-checker. First, a warm-up.

Example 1:
Determine an appropriate conclusion for the following puzzles:

(a) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will get a sunburn. Therefore, if I go swimming, then ___________

(b) (*another one from Lewis Carroll*) No ducks waltz. No officers ever decline to waltz. All my poultry are ducks. Therefore, ___________.

Solution:
(a) I will get a sunburn.

(b) My poultry are not officers.

1 General approach

The two most important things when trying to prove something are:

- Identify your **variables**
- State your **assumptions**: these becomes your **premises**

In identifying your variables, you will often need to translate sentences into math (like we did last class). The assumptions you make will generally follow from the description of the problem (so read carefully!).

Assumptions

Make sure you state all assumptions! If your assumption is incorrect (which will likely lead to an incorrect answer), then I'll be able to isolate the mistake and give you partial credit.

2 Drawing conclusions

With our variables and premises, we can now attempt to draw a conclusion. There are two main strategies for drawing conclusions.

2.1 Creating a truth table

Creating a truth table is the most intuitive way to determine if a conclusion is correct. However, it can be tedious, especially if you have a lot of variables and premises. After identifying your variables and premises, the general procedure is as follows:

1. Create a truth table for all variables
2. Create a column for each premise
3. Evaluate all premises using the True/False values in the truth table
4. Identify rows for which **all** premises are True.
5. Evaluate your theorem using the True/False values of your variables in the rows in which the premises are True.
6. If your conclusion is True when every premise is True, then you have confirmed the theorem.

x	y	z	p	q	C
T	T	T	T	F	
T	T	F	T	T	T
T	F	T	F	F	
T	F	F	T	T	T
F	T	T	F	F	
F	T	F	T	T	T
F	F	T	F	T	
F	F	F	F	F	

For example, a problem with 3 variables (x, y, and z) and 2 premises (p and q) and a conclusion C might have the truth table on the right. Of course, whether the premises or conclusion are true depends entirely on what they look like in terms of x, y and z (with the logical operators $\wedge$, $\vee$, and $\neg$).

Note that our conclusion C would be True because it evaluates to True for all the gray rows (where the premises all hold True). If this conclusion evaluated to False for any of those gray rows, then the conclusion would be false, and you would have to come up with a new conclusion. Luckily, this method looks a lot like an algorithm and can be used to check simple theorems. Philip's simple theorem prover is provided here, which you may use for quizlet 2 (hint!). The difficulty with the truth table method is that it works great if you already have an idea of the conclusion you are trying to deduce (since you need to evaluate it in the rows of your truth table). Therefore, we need another method for drawing up conclusions.

Can I automate this?

Automated theorem proving is a really hot topic these days. Although they are a little more complex than checking a truth table, software such as `coq` and `Isabelle` (to name a few) are frequently used to assist humans in constructing and checking proofs.

2.2 Reasoning by chaining premises

Another method for proving a conclusion is to chain the premises and use *rules of inference* to arrive at that conclusion. This one is a bit trickier, but useful if you have a lot of variables and premises. There is no recipe for how to chain together these premises.

Here are a few rules of inference. Note that this is not an exhaustive list but will suffice for our purposes. Given statements p, q and r:

Rule 1:

$$\begin{array}{r} p \\ p \implies q \\ \hline \therefore q \end{array}$$

Rule 2:

$$\begin{array}{r} p \implies q \\ q \implies r \\ \hline \therefore p \implies r \end{array}$$

Rule 3:

$$\begin{array}{r} \neg p \implies \neg q \\ \hline \therefore q \implies p \end{array}$$

Rule 4:

$$\begin{array}{r} \neg q \\ p \implies q \\ \hline \therefore \neg p \end{array}$$

These rules can be verified with a truth table. Once you have constructed a conclusion, you can then go back and check it with a truth table. Or you can use Philip's proof-checker to make life easier. :)

Therefore

Note that the $\therefore$ symbol is used to denote *therefore*.

Example 2:
It is not sunny this afternoon and it is colder than yesterday. We will go swimming only if it is sunny. If we do not go swimming, then we will take a canoe trip. If we take a canoe trip, then we will be home by sunset. Will we be home by sunset?

Solution:
Let the statements be labelled as follows:

- s: *it is sunny this afternoon*
- c: *it is colder than yesterday*
- w: *we will go swimming*
- p: *we will take a canoe trip*
- h: *we will be home by sunset*

We then have

step	statement (true)	explanation
1.	$\neg s \land c$	first sentence
2.	$\neg s$	simplification
3.	$w \implies s$	second sentence
4.	$\neg w$	rule 4
5.	$\neg w \implies p$	third sentence
6.	p	rule 1
7.	$p \implies h$	fourth sentence
$\therefore$	h	rule 1

Therefore, we will be home by sunset.

Let's now revisit the examples from the beginning of the lecture.

Example 3:

(a) We can label and translate the statements as follows:

s: you go swimming	(1) $s \implies t$
t: you stay in the sun too long	(2) $t \implies b$
b: you get a sunburn	(3) s (you went swimming)

Our deduction is then:

step	statement (true)	explanation
1.	s	premise 3
2.	$s \implies b$	premise 1 & 2 & rule 2
$\therefore$	b	rule 1

(b) This one is a bit trickier. Let X be the set of all beings. The statements can be translated as follows:

$d(x)$: x is a duck

$w(x)$: x waltzes

$o(x)$: x is an officer

$p(x)$: x is my poultry

(1) $\neg\exists x \in X\colon d(x) \wedge w(x)$

(2) $\forall x \in X,\ o(x) \implies w(x)$

(3) $\forall x \in X,\ p(x) \implies d(x)$

Our deduction is then:

step	statement (true)	explanation
1.	$\forall x \in X,\ \neg(d(x) \wedge w(x))$	negation of premise 1
2.	$\forall x \in X,\ \neg d(x) \vee \neg w(x)$	de Morgan on step 1
3.	$\forall x \in X,\ d(x) \implies \neg w(x)$	logically equivalent to step 2
4.	$\forall x \in X,\ p(x) \implies \neg w(x)$	step 3 and premise 3 using rule 2
$\therefore$	$p(x) \implies \neg o(x)$	step 4 with rule 3 on premise 2, then rule 2

The trickiest part is in step 3, when realizing that in $\neg d \vee \neg w$ is logically equivalent to the implication $d \implies \neg w$. Alternatively, you could arrive at this directly by reading the sentence *No ducks waltz* as *If something is a duck, then it does not waltz*, which is the same as $d \implies \neg w$.