Learning objectives:
- relate properties of functions to the mapping rule,
- apply product and addition rules to count sets without explicitly listing all items in the set,
- apply the pigeonhole principle to relate the cardinalities of two sets.

Last lecture, we talked about functions – more specifically, we looked at properties of functions (injective, surjective and bijective). These properties allowed us to characterize the relationship between the input set (domain) and the output set (codomain) of a function. Today, we want to start relating the cardinalities of these input and output sets, without having to explicitly define a function that maps the domain to codomain. First, an example!

Example 1:
Suppose you have a combination lock, which requires entering three numbers between \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\) and suppose it takes you one second to test a combination. How long will it take you to test every possible combination if:

(a) You know nothing about the digits.

(b) You are breaking into Philip’s locker, and you know he likes even numbers.

(c) You know that no number is repeated.

Solution:
(a) We have three numbers to pick, each of which has ten possibilities. So we have \(10 \times 10 \times 10\) possible combinations. Testing all of them would take 1000 seconds.

(b) Following a similar logic, there are only 5 even numbers to pick from, so \(5^3\) total combinations. It would only take 125 seconds (about two minutes) to break into Philip’s locker, so this is definitely do-able.

(c) This is a bit trickier. Say we pick a number \(n_1\) for the first digit. This leaves 9 possible digits to pick for the second \(n_2\). After picking a second, we only have 8 remaining digits to pick for the third digit \(n_3\). So there are a total of \(10 \times 9 \times 8\) possible combinations, which would require 630 seconds (about 10 minutes).
1 Rules

Recall our discussion about properties of functions. The **mapping rule** is a useful way to relate the sizes of sets, depending on how elements from a domain are mapped to a codomain. Given a function \( f : A \rightarrow B \),

- if \( f \) is injective, then \( |A| \leq |B| \),
- if \( f \) is surjective, then \( |A| \geq |B| \),
- if \( f \) is bijective, then \( |A| = |B| \).

This can be particularly useful if we know that a bijection exists between \( A \) and \( B \), because we then know that the cardinalities of \( A \) and \( B \) are the same.

**Product rule**  Let’s now formalize the intuition we used in the lock example for counting the number of lock combinations. Specifically, we used the **product rule** which says that if an event \( A \) can occur in \( m \) ways, and each possibility for \( B \) allows for exactly \( n \) ways for event \( B \), then the event \( A \) and \( B \) can occurs in \( m \times n \) ways.

**Addition rule**  If an event \( A \) can occur in \( m \) ways, and another **disjoint** event \( B \) can occur in \( n \) ways. Then the combined event \( A \) or \( B \) can occur in \( m + n \) ways. When applying the addition rule, it is important to identify events that are **disjoint** (there is no way they can occur at the same time).

If the two events \( A \) and \( B \) are not disjoint, then we need to subtract the number of events in which \( A \) and \( B \) both occur simultaneously.

Let’s now practice with some examples.

**Example 2:**
How many possible 6 character license plates are there if the first 3 characters must be letters, and the last 3 characters must be numbers?

**Solution:**
There are 26 possible letters for each of the first three characters, and 10 possible numbers for the last three characters, giving a total of \( 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000 \) possible license plates.

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*Three properties of functions*

Remember that a function \( f : A \rightarrow B \) is:

- **injective** if for every \( x, y \in A, \ x \neq y \) means \( f(x) \neq f(y) \).
- **surjective** if for every \( b \in B \), there is an \( a \in A \) with \( f(a) = b \).
- **bijective** if \( f \) is both injective and surjective.

*What if everyone in California bought a car?*

Hmmm, interesting question Piper. If everyone in the state of California (population \( \approx 39.5 \) million) bought a car, then we would run out of license plates with this model. This might be why California uses 7 characters on their license plates.
Example 3: How many possible passwords are there (passwords must be between 6 and 8 characters long), where each character is either an uppercase letter or a digit and passwords must contain at least one digit?

Solution: We know that passwords are either of 6, 7 or 8 characters. These events are all disjoint, so we can count the number of possibilities for each and then add them up. Let $p_n$ be the number of passwords of length $n$. There are 36 possibilities for each password character resulting from the 26 letters and 10 digits. The total number of password possibilities with $n$ characters is then $p_n = 36^n$. However, we have the restriction that passwords must contain at least 1 digit. This is violated if the passwords consist of only letters, for which there are $26^n$ possibilities. We need to subtract these possibilities, therefore giving a total of $p_n = 36^n - 26^n$ possibilities. To count passwords of length 6, 7 or 8, we have

$$\# \text{ passwords} = p_6 + p_7 + p_8 = (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8) \approx 2.68 \times 10^{12}.$$  

Principle of inclusion-exclusion In the last example, we could count the number of 6-, 7- or 8-character passwords by adding up the possible number of passwords of each. We could do this because each event was disjoint. But what if there is some overlap between several events?

We actually hinted at this earlier. If the events are not disjoint, then we need to subtract the number of events in which the events occurs simultaneously. This is know is the inclusion-exclusion principle because we are including (by adding) the number of possible events of each, but excluding (by subtracting) the number of events that occur simultaneously. For two sets $A$ and $B$, this can be written mathematically as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$
Example 4:
How many 5-bit strings start with 1 or end with 00? 16? 20? 24 or 32?

Solution:
There are two choices for each bit: either a 0 or a 1. The number of 5-bit strings that start with a 1 is $2^4$ since there are only 4 remaining bits to choose from after selecting a 1 for the first digit. By the same reasoning, the number of 5-bit strings that end with 00 is $2^3$ since we only have three bits left to choose after selecting 00 for the last two bits. However, there are common bit strings in these two options. Specifically, those strings which both start with a 1 and end with a 00. The number of common bit strings is $2^2$ since there are only two bits to choose from. The total number of bit strings that start with 1 or end with 00 is then $2^4 + 2^3 - 2^2 = 20$.

2 The pigeonhole principle

Remember when we proved that if there are $n+1$ Pipers and $n$ clamholes, then at least two Pipers will go for the same clamhole? This was actually the pigeonhole principle in disguise! Here is a generalized version of the pigeonhole principle, which is extremely useful in computer science.

**Theorem 1.** If $n$ objects are placed into $k$ boxes then there is at least one box contains $\geq \lceil \frac{n}{k} \rceil$ objects.

**Proof.** We use a proof by contradiction. Suppose $n$ objects are placed into $k$ boxes but there are no boxes containing $\geq \lceil \frac{n}{k} \rceil$ objects. In other words, all boxes contain $< \lceil \frac{n}{k} \rceil$ objects. We can rewrite this as: all boxes contain $\leq \lceil \frac{n}{k} \rceil - 1$ objects. With $k$ boxes, the total number of objects is then $\leq k (\lceil \frac{n}{k} \rceil - 1)$. But $\lceil \frac{n}{k} \rceil < \frac{n}{k} + 1$ so overall there would be less than $k (\lceil \frac{n}{k} \rceil + 1) - 1$ objects, which is a contradiction.

Example 5:
Given 35 people in a single room, show that there are at least three people born in the same month.

**Solution:**
The following direct proof uses the pigeonhole principle along with the fact that there are 12 months in a year.

**Proof.** By the pigeonhole principle, there are 35 people and 12 months for each person’s birthday. Therefore, at least $\lceil \frac{35}{12} \rceil = 3$ people are born in the same month.