This **two-hour** exam is **closed book** and **closed notes** and consists of **one double-sided page**. Some notes are provided on the second page.

Pick any 4 **of the following** 5 **questions** (there are 6 questions in this practice midterm, so there's an extra one for practice) to solve in the blue exam booklets. You **must circle** (on this sheet) which questions you wish to be graded. Please don't make me guess! You must **show all your work** to receive full credit. Simply stating the answer is not enough.

Please sign the honor code: *I have neither given nor received unauthorized aid on this assignment.*

Problem 1

Are the two statements $(p \implies q) \implies r$ and $(p \wedge q) \implies r$ logically equivalent?

Problem 2

Let S be the set of all people who have ever lived. Let $g(x, y)$ be the predicate that x is the grandmother of y, for $x, y \in S$. Let $c(x, y)$ be the predicate that x and y are cousins, for $x, y \in S$. Translate the following propositions and predicates into a mathematical representation:

(a) All people have at least two grandmothers.

(b) Every pair of cousins share a grandmother.

(c) None of person x's cousins are grandmothers.

Problem 3

Let $A = \{1, 2, 3, 4, 5\}$. Describe the following sets in roster notation:

(a) $\{x \in \mathbb{Z} \colon \exists y \in A,\ x = y^2 + 1\}$

(b) $\{x \in \mathbb{Z} \colon \exists y \in A,\ y = x^2 + 1\}$

Express the following set using set builder notation:

(c) The set of all *pairs* of integers such that the numbers in a pair are non-zero, have opposite signs, and that the absolute value of one of them is the square of the absolute value of the other. You can use the notation (m, n) to represent the two numbers m and n as a pair.

Problem 4

Prove using induction that $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ where F_n is the n^{th} Fibonacci number. Recall that Fibonacci numbers are defined recursively by

$$F_n = F_{n-1} + F_{n-2}, \quad \text{with } F_0 = 0, \text{ and } F_1 = 1.$$

Problem 5

Prove that $\sqrt{2}$ is irrational. *Hint: proof by contradiction.*

Problem 6

Prove using induction that the following algorithm outputs $n(n+1)/2$ for all $n \in \mathbb{N}$.

```
func(n)

  input: n ∈ ℕ
  output: some integer, which we will prove is n(n+1)/2
  if n == 1      # base case
      return 1
  else   # recursive case
      return n + func(n − 1)
```

Rules of inference

Rule 1:

$$\begin{array}{r} p \\ p \implies q \\ \hline \therefore q \end{array}$$

Rule 2:

$$\begin{array}{r} p \implies q \\ q \implies r \\ \hline \therefore p \implies r \end{array}$$

Rule 3:

$$\begin{array}{r} \neg p \implies \neg q \\ \hline \therefore q \implies p \end{array}$$

Rule 4:

$$\begin{array}{r} \neg q \\ p \implies q \\ \hline \therefore \neg p \end{array}$$

Set symbols

symbol	meaning	explanation
$A \cap B$	set intersection	set containing all elements which are elements of both A and B
$A \cup B$	set union	set containing all elements which are elements of A or B or both
$\overline{A}$	complement	set of everything which is not an element of A
$A \setminus B$	set-minus	set containing all elements of A which are not elements of B
$\lvert A \rvert$	cardinality	number of elements in A

Simplifying compound propositions

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad \text{de Morgan}$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad \text{de Morgan}$$
$$\neg\neg p \equiv p \qquad \text{double negation}$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \qquad \text{distributive}$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \wedge r) \qquad \text{distributive}$$
$$p \implies q \equiv \neg q \implies \neg p \qquad \text{contrapositive}$$

Negating quantifiers

$$\neg \forall x \in A,\ p(x) \equiv \exists x \in A \colon \neg p(x)$$
$$\neg \exists x \in A \colon p(x) \equiv \forall x \in A,\ \neg p(x)$$