Please print your name:

This two-hour exam is closed book and closed notes.

Pick any 4 of the following 5 questions to solve directly on this exam. You must circle which questions you wish to be graded - please don’t make me guess!

You must show all your work to receive full credit. Simply stating the answer is not enough.

Please feel free to grab an extra (blank) sheet of paper from your notebook if you need any additional room to show your work.

Please sign the honor code: I have neither given nor received unauthorized aid on this assignment.

Notes

<table>
<thead>
<tr>
<th>symbol</th>
<th>meaning</th>
<th>explanation</th>
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</thead>
<tbody>
<tr>
<td>$A \cap B$</td>
<td>set intersection</td>
<td>set containing all elements which are elements of both $A$ and $B</td>
</tr>
<tr>
<td>$A \cup B$</td>
<td>set union</td>
<td>set containing all elements which are elements of $A$ or $B or both</td>
</tr>
<tr>
<td>$\overline{A}$</td>
<td>complement</td>
<td>set of everything which is not an element of $A</td>
</tr>
<tr>
<td>$A \setminus B$</td>
<td>set-minus</td>
<td>set containing all elements of $A$ which are not elements of $B</td>
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<tr>
<td>$</td>
<td>A</td>
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</table>

Theorem 1. The sum of the vertex degrees equals twice the number of edges in a graph: $\sum_{v \in V} \deg(v) = 2|E|$.

Theorem 2. A graph with maximum degree at most $k$ can be colored with $(k + 1)$ colors.

Theorem 3. A tree with $n$ vertices has $n - 1$ edges.

Theorem 4. A full $k$-ary tree is a $k$-ary tree in which every internal vertex (which includes the root, unless it is the only vertex) has $k$ children. Here are a few properties of full $k$-ary trees:

(a) The number of vertices $n$ in a full $k$-ary tree with $i$ internal vertices is $n = ki + 1$.

(b) The number of leaves $\ell$ in a full $k$-ary tree with $i$ internal vertices is $\ell = i(k - 1) + 1$.

The number of ways to choose $k$ items out of $n$ is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} = C(n,n-k).$$

The number of ways to permute $k$ items out of $n$ is

$$P(n,k) = \frac{n!}{(n-k)!} = C(n,k) \cdot k!.$$ 

The number of ways to pick the location of $n$ stars, or $k - 1$ bars, from $n + k - 1$ positions is

$$\binom{n + k - 1}{n} = \binom{n + k - 1}{k - 1}.$$
**Problem 1**

During a team workout of **nine** athletes, the trainer must create a schedule for each exercise. There are **eight** exercises, each of which can be performed by **two or three** athletes at any given time. The assignment of athletes to each of the eight exercises is given below:

- $E_1$: Nadia, Eduardo, Charlie
- $E_2$: Nadia, Diane, Eve
- $E_3$: Eduardo, Kento
- $E_4$: Greta, Diane, Hamza
- $E_5$: Greta, Jiao, Eve
- $E_6$: Jiao, Kento
- $E_7$: Jiao, Diane
- $E_8$: Eduardo, Kento, Eve

Clearly, no two athletes can be doing different exercises at the same time. We need to help the trainer by determining the minimum number of time slots required such that all athletes complete all their assigned exercises.

(a) Recast this problem as a question about coloring the vertices of a particular graph. Draw the graph and explain what the vertices, edges, and colors represent.

(b) Show a coloring of this graph using the fewest possible colors. What athlete-exercise schedule does this imply?
Problem 2

A tournament graph \( G = (V, E) \) is a directed graph such that there is either an edge from \( u \) to \( v \) or an edge from \( v \) to \( u \) for every distinct pair of vertices \( u \) and \( v \). The vertices represent players and an edge \( u \rightarrow v \) indicates that player \( u \) beats player \( v \).

Consider the "beats" relation implied by a tournament graph. Note that a player never beats themselves. Determine whether the "beats" relation is

(a) reflexive?
(b) symmetric?
(c) transitive?
Problem 3

A bipartite graph is a graph $G$ in which the vertices can be divided into two disjoint sets $X$ and $Y$, such that $V(G) = X \cup Y$ and $X \cap Y = \emptyset$, and each edge in $G$ has one vertex in $X$, and one vertex in $Y$.

Prove using induction that every tree with at least two vertices is a bipartite graph. *Hint: proof by induction on the number of vertices.*
Problem 4
Suppose you are writing a program that schedules jobs to run on a high-performance computing (HPC) cluster. In how many ways can you assign six different jobs to three compute nodes so that every node is assigned at least one job?
Problem 5

(a) How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

(b) How many integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 + x_5 = 13$ with all $x_i \geq 0$ ($1 \leq i \leq 5$)?