Please complete this worksheet by October 29th, 2021 by 11:59pm.

Once you upload a picture of your work (here), the solutions will become available so you can study for the weekly quizlet, which may draw one problem from this week’s worksheets.

Problem 0: some theory

The past few topics may have seemed a bit disconnected, but actually we can connect them to graph theory (we’ll also see how functions can be connected to counting techniques next week). Let me introduce the idea of an isomorphism.

Definition 1. An isomorphism between two graphs $G$ and $H$ is a bijection $f: V(G) \rightarrow V(H)$ such that

\[(u, v) \in E(G) \iff (f(u), f(v)) \in E(H)\]

It’s a good idea to say this out in words: two graphs are isomorphic if we can find a relabeling of the vertices of the two graphs, and an edge in the first graph is an edge between the relabeled vertices in the second graph. Note that this ‘relabeling’ refers to the bijective function $f$ between the vertices of the two graphs.

Let’s look at an example.

These two graphs are isomorphic because we can define a bijective function between the vertices that preserves the edges:

\[f(a) = x \quad f(b) = z \quad f(c) = v \quad f(d) = y \quad f(e) = u.\]

How do we check if two graphs are isomorphic? If you can find a bijection (a relabeling) between the vertices of the two graphs and the connections between edges are preserved, then the graphs are isomorphic (by the definition above). However, we also have some quicker ways to tell whether two graphs are not isomorphic: we can check some properties of the two graphs. Two graphs are isomorphic if:

- they have the same number of vertices and edges,
- every vertex adjacent to $v$ in the first graph will be mapped to a vertex adjacent to $f(v)$ in the second graph. This means $v$ and $f(v)$ must have the same degree.

For example, if two graphs have a different number of vertices, then they are not isomorphic. Also, if one graph has a vertex of degree 4 (for example), but the other graph does not, then they are not isomorphic. One last check we can do is on the cycles. If one graph has a cycle of length $k$ but the second graph does not, then they are not isomorphic. If these checks pass, then you should try to come up with a bijective function that relabels the vertices!

Finally, isomorphism is an equivalence relation! All graphs that are isomorphic to one another are in the same equivalence class.
Problem 1
For each function below from $\mathbb{Z}$ to $\mathbb{Z}$, state whether the function is surjective, injective or bijective.

(a) $f(n) = n - 1$
(b) $f(n) = n^2 + 1$
(c) $f(n) = n^3$

Problem 2
The floor and ceiling functions come up frequently in computer science. Their domain is the real numbers and their codomain is the integers. For example, $\lfloor x \rfloor$ (the floor of $x$) returns the largest integer less than or equal to $x$ whereas $\lceil x \rceil$ (the ceiling of $x$) returns the smallest integer greater than or equal to $x$.

(a) What is $\lfloor -\sqrt{2} \rfloor$?
(b) What is $\lceil -\sqrt{2} \rceil$?
(c) Is the ceiling function surjective? Please explain.
(d) Is the floor function injective? Please explain.