We are going to solve some problems together in class - nothing to submit :)

**Problem 1: stars and bars**

As a warm-up, try to count the number of ways you can distribute the four candies I have given each group to three group members. Assume the candies are identical and each of the three group members can have 0, 1, 2, 3 or 4 candies. Also assume that all four candies must be given away.

Now, let’s extend the candy distribution problem. In how many ways can 100 identical candies be distributed to 20 students? Assume that every student must get at least one candy and all 100 candies must be given away.

*Hint:* first give every student a candy and then think about how many non-negative integer solutions there are to the equation \(x_1 + x_2 + \cdots + x_{20} = 80\) (solving for the 20 \(x_i\)'s). Extra hint: see this section in Levin’s DMOI.

**Solution:** Since every student must get at least one candy, then first give one candy to every student. This leaves 80 extra candies to distribute to 20 students. As per the hint, let the extra candies each student receives be represented by \(x_i\) (\(1 \leq i \leq 20\)), so we need to count the number of non-negative integer solutions to \(x_1 + x_2 + \cdots + x_{20} = 80\) (some of the \(x_i\)'s can be 0 or 80), which is similar to our warm-up problem. The key thing to ask here is (1) what are we choosing (and how many)? and (2) how many options do we have for each choice? Well, we have 80 candies to choose from (so 80 assignments), and we need to pick how they get divided among 20 students. However, we only really have 19 students to pick from, because once we pick those, then the number of candies given to the 20\(^{th}\) student must satisfy the fact that the total must be 80. Therefore, there are 80 + 19 choices for each of the 80 assignments. This is a total of \(\binom{80+19}{80} = \binom{99}{80}\) ways.

This is known as the **stars and bars** technique in which we have \(n\) objects (or stars \(\star\)) that we want to assign to \(k\) positions (separated by \(k-1\) bars \(|\)). For example, take \(n = 12\) and \(k = 5\) (12 candies and 5 students):

\[
\{\star\star\star\star | \star | \star\star | \star\star | \star\star\}.
\]

We really need to pick the position for the \(n\) stars out of the \(n+k-1\) total possible positions (remember to consider the position of both stars *and* bars). You can see this either as picking the position of the \(n\) objects (stars), or the \(k-1\) bars and remember a nice property of the binomial coefficient:

\[
\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.
\]

In our problem, \(n = 80\) and \(k = 20\) after we’ve given one candy to every student at the beginning.

**Problem 2**

(a) How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

(b) In how many ways can you arrange 6 red balls, 5 green balls, 3 blue balls, 1 white ball and 1 black ball in a row? Assume the balls are identical other than color.

(c) How many bit strings of lengths 10 have (i) exactly three 0s? (ii) at least seven 1s? (iii) exactly three 0s or start with a 1?
Solution:

(a) There are 13 values to pick from for the value of the pair. There are 2 suits to pick from 4 possible suits for the suits of the pair: \( \binom{4}{2} \). If we are not allowed to have a 3-of-a-kind or 4-of-a-kind, then the remaining three cards need to have a value distinct from the value of the pair: there are 12 options and we need to pick 3 of these, so \( \binom{12}{3} \). There are 4 suit options for each of these three cards, so the total 5-card hands with a single pair is \( 13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3 \).

(b) If the objects are distinct, there are a total of \( 6 + 5 + 3 + 1 + 1 = 16 \) balls, which can be arranged in \( 16! \) possible ways. We can permute the 6 red balls without changing the overall arrangement of all 16 balls, which is \( 6! \) permutations. So we need to divide the result by \( 6! \). The same is true for the 5 green and 3 blue ones. Therefore, the number of arrangements is \( \frac{16!}{6!5!3!} \).

(c) For part (i) we need to pick the positions of the 3 zeros out of 10 possible positions, of which the order doesn’t matter, i.e. pick 3 positions out of 10, so \( \binom{10}{3} = 120 \). For part (ii), we can use the addition rule for the number of length-10 bit strings with exactly 7, 8, 9 or 10 ones: \( \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = 176 \). For part (iii), we will use the principle of inclusion-exclusion. There are \( \binom{10}{3} \) length-10 bit strings that have three zeros, and \( 2^9 \) length-10 bit strings that start with a 1 (after fixing the first bit at 1, there are 9 more to pick, each with 2 options). The number of length-10 bit strings that start with a 1 and have three 0s is equal to \( \binom{9}{3} \) since after fixing the first bit at 1 we can pick 3 of the remaining 9 bits to be 0s. Using the principle of inclusion-exclusion, the total for part (iii) is then \( \binom{10}{3} + 2^9 - \binom{9}{3} \) possible strings.